"Custom Collatz Sequence: Modular Reduction, Isopath Orbits and Analysis of Odd and Even Steps"

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Date: July 9, 2025

Abstract

One of the problems of elementary mathematics, still unsolved, is the Collatz conjecture (after the mathematician Lothar Collatz). It concerns the following function between natural numbers:
f: N →N

$$f(N)=\begin{cases} N/2 \text{ if N is even} \\ 3*N+1 \text{ if N is odd} \end{cases}$$

Iterating the function several times after a certain number of iterations gives the number 1 and continuing the iterations gives the cycle 1–4–2–1. Then the initial number N, in subsequent iterations, is multiplied by 3 (adding 1) and divided by 2 so that it is a succession of odd steps (when multiplying by 3 and adding 1) and even passes (when dividing by 2) which, after a certain total number of even passes ***bi*** and a certain total number of odd steps ***ai***, converges to 1 according to the relation where is the final residue which is between $\frac{3^{a_i}*N*Res_i}{2^{b_i}} = 1$ Res_i**1.1** and **1.25** for reference **[2]** and depends on the sum of 1 in the iterations in which there is the odd step **(i.e. 3*N+1)** and therefore we will have . $\frac{3^{a_i}*N}{2^{b_i}} < 1$

In this paper we present a customized sequence of the Collatz function, formulated on a modular basis and oriented to reduce the number of iterations required for convergence. The approach is based on the classification of the natural numbers module 4, concentrating the analysis only on the odd numbers congruent with 1 and 3 module 4. Thanks to algorithmic coding and a diagrammatic structure, the method allows for simplified tracking of odd and even steps, with obvious computational advantages. The comparison with the canonical sequence of Collatz, applied to the number N=27, shows a significant reduction in iterations (from 111 to 21), while maintaining the final convergence. The introduction of the concept of isopath orbits strengthens the efficiency of the model and opens theoretical perspectives for a constructive proof of the famous conjecture.

1 Introduction

In my personalized and faster sequence of Collatz, only odd numbers are used (after a possible initial number that can also be even mod. 4) only odd numbers ≡ 1 mod. 4 and ≡ 3 mod. 4 because they represent 50% of a group of consecutive integers and of these half are ≡ 1 mod. 4 which always have a subsequent number in the sequence less than the previous one and the other half are ≡ 3 mod. 4 which, on the basis of **Important Consideration 5.1** and **Note 6.2 of reference [1],** have a probability of **10.9375% (out of 25%)** of having a subsequent number in the succession that is lower than the previous one. So in conclusion, the sequence I customize in each iteration has a probability of **35.9375% (out of 50%)** of having a subsequent number in the sequence that is less than the previous one and a probability of **14.0625% (out of 50%)** of having a subsequent number in the sequence that is greater than the previous one.
Furthermore, from the first iteration the total of even passages (which I indicate with ***bi)*** is always greater than the total of odd passages (which I indicate with ***ai),*** i.e. we have ***bi > ai*** and in subsequent iterations we have that ***bi*** increases more than ***ai*** and it also happens that the difference between the total of even

passes and the total of odd steps always increases by 1 or 2 units in each iteration and this produces in subsequent iterations a difference between ***bi*** and ***ai*** that always increases.

2 Custom Collatz Sequence

Important note: The following sequence I customized applies to all integers **N other than (2^p-1)/3 with p>=2 and with p even integer i.e. the set {1 – 5 – 21 – 85 – 341 – 1365 - ....}. In fact, these (2^p-1)/3 odd and integers satisfy Collatz's conjecture and immediately converge to 1 after only one odd step and p even steps** (insert this condition in the Python program**).** In fact, applying the odd passage 3*N+1 to them we obtain the number **2^p** which after p even passages converges to 1.

Here is a **textual representation** of the **custom Collatz sequence** applied to all integers **N ≠(2^p-1)/3 with p even integer,** ready to be transformed into a python program where you can test the veracity of the proof: Here is a **textual representation** of the **custom Collatz Sequence,** ready to be transformed into a python program where you can test the veracity of the proof:

🔁 **Custom Collatz Succession Flowchart**

1. **Start N**
2. **N Modulus Control**:
   - If N≡ 0 mod 4 → write **N=2^p*(1+2*k)** for each k and p natural numbers → **Go to step 1;**
   - If N≡ 2 mod 4 → write **N=4*(1+2*k)+2** for each integer k>=0 → **Go to step 2;**
   - If N≡ 1 mod 4 → write **N=4*(1+2*k)+1** or **N=4*(2*k)+1** for each k>=0 integer → **Go to step 3 or 4 if the quotient of dividing N by 4 is odd (1+2*k) or even (2*k) respectively;**
   - If N≡ 3 mod 4 → write **N+1=2^p*(1+2*k)** from which **N=2^p*(1+2*k)-1** with **k≥0** and **p≥2 integers** → **Go to step 5 or 6 if the values of k and p have the same parity or different parity.**

🔹 **Point 1: N≡ 0 mod 4**

1. if the initial number **N of the sequence is even,** i.e. **N ≡ 0 mod 4**, we can write **N=2^p*(1+2*k)** for every k and p natural numbers. So its next in the sequence (after p even steps) is the odd number **1+2*k** which **if k is even will be 1+2*k ≡ 1 mod 4** while **if k is odd it will be 1+2*k ≡ 3 mod 4** and therefore in this iteration (from N to 2*k+1) the odd steps are *a=0 while the even steps are b=p* **(the difference between odd and even is increased by p unit)**;

Summary Point 1: N≡ 0 mod 4

- Write N=2^p·(1+2k)
- After **p even passes**, the next of N is **1+2k**
- If k is equal → 1+2k≡ 1 mod 4
- If odd k → 1+2k≡ 3 mod 4
- ✅ Odd passes: a=0 , even: b=p, difference b-a=p.

🔹 **Point 2: N≡ 2 mod 4**

2. if the initial number **N of the sequence is even,** i.e. **N ≡ 2 mod 4** you can write **N=4*k+2** for every integer k>=0 then its next in the sequence (after 1 even step) is the odd number **2*k+1** which will be **2*k+1 ≡ 1 mod 4 if k is even** instead **it will be 4*k+3 ≡ 3 mod 4 if k is odd** and therefore in this iteration (from N to 2*k+1) the odd steps are a=0 while the even steps are b=1 **(the difference between odd and even is increased by 1 unit)**;

Summary Point 2: N≡ 2 mod 4

- Write N=4*k+2
- After **1 even pass**, the next one of N is **2k+1**
- If even k → 2k +1≡ 1 Mode 4
- If odd k → 2k +1≡ 3 mod 4
- ✅ Odd passes: a=0, even: b=1, difference b-a=1.

🔹 **Point 3: N≡ 1 mod 4, with N=4·(1+2k)+1 where the quotient between N and 4 is 1+2*k odd**

3. if the initial number **N of the sequence is odd,** i.e. **N ≡ 1 mod 4**, we can write **N=4*(1+2*k)+1** for every integer k>=0 (so the quotient 1+2*k is odd) then its next in the sequence (after an odd step and two even steps) is the even number **3*(1+2*k)+1** which is isopath to the **odd number 1+2*k** (therefore **"change of orbit"** and there is no longer the odd passage) and the sequence will continue with the odd number **1+2*k** so that in this iteration (from N to 2*k+1) the odd passages are *a=0 while the even passages are b=2*. In addition, the number **1+2*k if k is even will be 1+2*k ≡ 1 mod 4** while **if k is odd it will be 1+2*k ≡ 3 mod 4** (**the difference between odd and even is increased by 2 units)**;

Summary Point 3: N≡ 1 mod 4, with N=4·(1+2k)+1 where the quotient between N and 4 is 1+2*k odd

- After **0 odd pass + 2 even passes**, the next of N is **1+2k**
- If k is equal → 1+2k≡ 1 mod 4
- If odd k → 1+2k≡ 3 mod 4
- ✅ Odd passes: a= 0, even: b=2, difference b-a=2.

🔹 **Point 4: N≡ 1 mod 4, with N=4·(2k)+1 where the quotient between N and 4 is 2*k even**

4. if the initial number **N of the sequence is odd,** i.e. **N ≡ 1 mod 4** and you can write **N=4*(2*k)+1** for every integer k>=0 (so the quotient 2*k is even) then its next in the sequence (after an odd step and two even steps) is the odd number **3*(2*k)+1=6*k+1** which is odd so in this iteration (from N to 6* k+1) odd steps are *a=1 while even steps are b=2*. In addition, the number **6*k+1 if k is even will be 6*k+1 ≡ 1 mod 4** while **if k is odd it will be 6*k+1 ≡ 3 mod 4 (the difference between odd and even is increased by 1 unit)**;

Summary Point 4: N≡ 1 mod 4, with N=4·(2k)+1 where the quotient between N and 4 is 2*k even

- After **1 odd pass + 2 even passes**, the next of N is **6k+1**
- If k evens → 6k+1≡ 1 mod 4

- If k odd → 6k+1≡ 3 mod 4
- ✅ Odd steps: a = 1, even: b = 2, difference b-a=1.

🔹 **Step 5: N≡ 3 mod 4 with k and p both even or both odd and with k≥0** and **p≥2**

5. if the initial number **N of the sequence is odd, i.e. N ≡ 3 mod 4,** we can write **N+1=2^p*(1+2*k)** from which **N=2^p*(1+2*k)-1** with **k≥0** and **p≥2 both odd or both even**, then its next in the sequence (after p-1 odd steps and p+1 even steps) is the odd number **[3^(p-1)*(1+2*k)-1]/2** and therefore the odd steps are *a=p-1 while even passes are b=p+1* **(the difference between odd and even is increased by 2 units).** (see **theorems 2.1 – 2.2 – 2.3** and **note 2.1 reference [1]**)

Summary Point 5: N≡ 3 mod 4 with k and p both even or both odd and with k≥0 and **p≥2**

- Write N+1=2^p· (1+2k) → N=2^p· (1+2k)−1
- After **p−1 odd passes** and **p+1 even passes**, the next of N is **[3^(p−1) · (1+2k)−1]/2**
- ✅ a=p−1, b=p+1, difference b-a=2.

🔹 **Point 6: N≡ 3 mod 4 with k even** and **p odd** or **with k odd** and **p even and with k≥0** and **p≥2**

6. if the initial number **N of the sequence is odd, i.e. N ≡ 3 mod 4** and you can write **N+1=2^p*(1+2*k)** from which **N=2^p*(1+2*k)-1 with k even and p odd or with k odd and p even** then its next in the sequence (after p odd steps and p+1 even steps) is the odd number **[3^(p)*(1+2*k)-1]/2** and therefore the odd passages are *a=p while the even passages are b=p+1* **(the difference between odd and even is increased by 1 unit)**. (see **theorems 2.1 – 2.2 – 2.3** and **note 2.1 of reference [1]**)

Summary Point 6: N≡ 3 mod 4 with k even and **p odd** or **with k odd** and **p even and k≥0** and **p≥2**

- Write N+1=2^p· (1+2k) → N=2^p· (1+2k)−1
- After **p odd steps** and **p+1 even passes** , the next of N is **[3^p · (1+2k)−1]/2**
- ✅ a=p, b=p+1, difference b-a=1.

4 Example of a custom Collatz sequence.

In the following **Table 1** as an example there is the sequence of the initial number N=4*6+3=27 with the succession of odd numbers of type 3 and 4. This number is known to have a particularly long trajectory in canonical succession. In column 1 is the number of iterations. In column 2 there is the sum of the total odd steps with the total even passes up to the iteration just performed. In column 3 there is the sequence of the number N. In column 4 the single number indicates p i.e. the multiplicity of the 2 of N+1 when the number is **N ≡ 3 mod 4 (points 5 or 6 of the sequence)** or the symbol "0d2p" when the number is **N ≡ 1 mod 4 (point 3 of the sequence)** for which there are 2 even numbers and 0 odd numbers or the symbol "1d2p" when the number is **N ≡ 1 mod 4 (point 4 of the sequence) sequence)** for which there are 2 even numbers and 1 odd numbers. In column 5 there are the values of (1+2*k) in the case that the number **N ≡ 3 mod 4 (points 5 or 6 of the sequence)**. In column 6 there are the values k (for which k is odd or even) in the case that the number **N ≡ 3 mod 4 (points 5 or 6 of the sequence).** Column 7 specifies the point in the **Custom Collatz Sequence Flowchart out of** the 6 overall. In column 8 there is the total number of odd passages and in column 9 there is the total number of even passages since the beginning of the sequence. In column 10 are the values of the difference between the total even passes

and the total odd passes. Column 11 shows the ratio of the total odd steps to the number of even steps in the iteration. In column 13, the 'YES' indicates that the number of the transformed sequence is less than the initial number N.
In the following **Table 2** there is the canonical Collatz sequence of the initial number N=4*6+3=27 with the succession of odd or even single steps. In column 1 there is the number of iterations, in column 2 there is the sequence of the number N, in column 3 there is the total number of odd steps and in column 4 there is the total number of even steps since the beginning of the sequence.

Table 1

Customized Collatz sequence using odd numbers only N≡1 mod.4 and N≡3 mod.4											
tions	step	N	Periodicity p or odd and even numbers	Value of w=1+2*k	k odd or even	Transformation of N with respect to the 6 points of the sequence	Total odd	Total even	Difference between even and odd steps	Average between odd and even	N'<N
0		27					0	0	0		
1	5	31	2	7	3	Punto 6	2	3	1	0,666666667	
2	16	121	5	1	0	Punto 6	7	9	2	0,777777778	
3	19	91	1d2p			Punto 4	8	11	3	0,727272727	
4	24	103	2	23	11	Punto 6	10	14	4	0,714285714	
5	31	175	3	13	6	Punto 6	13	18	5	0,722222222	
6	40	445	4	11	5	Punto 6	17	23	6	0,739130435	
7	42	111	0d2p			Punto 3	17	25	8	0,68	
8	51	283	4	7	3	Punto 6	21	30	9	0,7	
9	56	319	2	71	35	Punto 6	23	33	10	0,696969697	
10	68	607	6	5	2	Punto 5	28	40	12	0,7	
11	78	769	5	19	9	Punto 5	32	46	14	0,695652174	
12	81	577	1d2p			Punto 4	33	48	15	0,6875	
13	84	433	1d2p			Punto 4	34	50	16	0,68	
14	87	325	1d2p			Punto 4	35	52	17	0,673076923	
15	89	81	0d2p			Punto 3	35	54	19	0,648148148	
16	92	61	1d2p			Punto 4	36	56	20	0,642857143	
17	94	15	0d2p			Punto 3	36	58	22	0,620689655	SI
18	102	13	4	1	0	Punto 5	39	63	24	0,619047619	SI
19	104	3	0d2p			Punto 3	39	65	26	0,6	SI
20	108	1	2	1	0	Punto 5	40	68	28	0,588235294	SI
21	111		1d2p			Punto 4	41	70	29	0,585714286	SI

Table 2

Canonical succession of Collatz			
Iterations	N	Total odd	Total even
0	27		
1	82	1	0
2	41	1	1
3	124	2	1
4	62	2	2
5	31	2	3
6	94	3	3
7	47	3	4
8	142	4	4
9	71	4	5
10	214	5	5
11	107	5	6
12	322	6	6
13	161	6	7
14	484	7	7
15	242	7	8
16	121	7	9
17	364	8	9
18	182	8	10
19	91	8	11
20	274	9	11
21	137	9	12
22	412	10	12
23	206	10	13
24	103	10	14
25	310	11	14
26	155	11	15
27	466	12	15
28	233	12	16

Canonical succession of Collatz			
Iterations	N	Total odd	Total even
29	700	13	16
30	350	13	17
31	175	13	18
32	526	14	18
33	263	14	19
34	790	15	19
35	395	15	20
36	1186	16	20
37	593	16	21
38	1780	17	21
39	890	17	22
40	445	17	23
41	1336	18	23
42	668	18	24
43	334	18	25
44	167	18	26
45	502	19	26
46	251	19	27
47	754	20	27
48	377	20	28
49	1132	21	28
50	566	21	29
51	283	21	30
52	850	22	30
53	425	22	31
54	1276	23	31
55	638	23	32
56	319	23	33
57	958	24	33

Canonical succession of Collatz			
Iterations	N	Total odd	Total even
58	479	24	34
59	1438	25	34
60	719	25	35
61	2158	26	35
62	1079	26	36
63	3238	27	36
64	1619	27	37
65	4858	28	37
66	2429	28	38
67	7288	29	38
68	3644	29	39
69	1822	29	40
70	911	29	41
71	2734	30	41
72	1367	30	42
73	4102	31	42
74	2051	31	43
75	6154	32	43
76	3077	32	44
77	9232	33	44
78	4616	33	45
79	2308	33	46
80	1154	33	47
81	577	33	48
82	1732	34	48
83	866	34	49
84	433	34	50
85	1300	35	50
86	650	35	51

Canonical succession of Collatz			
Iterations	N	Total odd	Total even
87	325	35	52
88	976	36	52
89	488	36	53
90	244	36	54
91	122	36	55
92	61	36	56
93	184	37	56
94	92	37	57
95	46	37	58
96	23	37	59
97	70	38	59
98	35	38	60
99	106	39	60
100	53	39	61
101	160	40	61
102	80	40	62
103	40	40	63
104	20	40	64
105	10	40	65
106	5	40	66
107	16	41	66
108	8	41	67
109	4	41	68
110	2	41	69
111	1	41	70

It should be remembered that the "change of orbit" in the canonical sequence (Table 2) is obtained in these 2 cases:

a. when at the same time the preceding number N is even and (N-1)/3 is an odd number so that the next number of N is the number (N-1)/3;
b. when at the same time the preceding number N is odd and (N-1)/4 is an odd number so that the next number of N is the number (N-1)/4.

Comparing the two tables shows:

1. the number 27 converges to 1 with 111 total passages, therefore with 111 iterations in Table 2, while in Table 1 the number of iterations is 21;
2. The computational convenience **of the custom sequence is emphasized, emphasizing not only the reduction in the number of iterations but also the reduction in complexity;**
3. in column N of Table 1 there are not all the even numbers of column N of Table 2 since Table 1 is formed by the succession of only odd numbers (after a possible initial number which can also be even mod. 4);
4. the numbers in light blue in column N of Table 1 are isopath with respect to the numbers in light blue in column N in Table 2; therefore they correspond in that there is the "change of orbit" based on the following points 5. and 6. of this list;
5. in Table 1 all the odd numbers in light blue in column N that correspond to point 3 of the custom sequence are not present in column N of Table 2 because they derive from "orbit change" case a). They are in Table 2 the numbers 334 – 244 – 46 – 10 which correspond to the numbers in Table 1 colored in light blue respectively 111 – 81 – 15 – 3 as the latter are isopaths to them. In fact (334 – 1)/3= 111 - (244 – 1)/3= 81 - (46 – 1)/3= 15 - (10 – 1)/3= 3;
6. in Table 1 all the odd numbers in light blue in column N that correspond to point 5 of the custom sequence are not present in column N of Table 2 because they derive from "orbit change" case b). They are in Table 2 the numbers 2429 – 3077 – 53 – 5 which correspond to the numbers in Table 1 colored in light blue respectively 607 – 769 – 13 – 1 as the latter are isopaths to them because (2429 -1)/4=607 - (2429 -1)/4=607 - (3077 -1)/4=769 - (53 -1)/4=13 - (5 -1)/4=1.

5 Interpretation of Results

The example of the initial number N=27 clearly highlights the effectiveness of the personalized Collatz sequence compared to the canonical. In fact, a **significant difference emerges in the number of iterations**: 21 against 111. This reduction is the result of optimized trajectory management, based on:

- **Classification module 4**, which allows you to immediately identify the type of orbit to be applied;
- **Elimination of the intermediate even steps**, which in canonical succession slow down the path towards unity;
- **Using the concept of isopath**, which allows you to replace groups of numbers with simpler equivalents without losing structural information:

Moreover, the **increasing difference between the total number of odd and even passes** demonstrates a structural trend that favors convergence: the constant increase of 1 or 2 units of

bi−ai for each iteration i not only speeds up the process, but suggests that the acceleration is systemic and not random.

Theoretical implications: If the custom sequence maintains this efficiency for arbitrary values of N, new scenarios open up for a constructive proof of the Collatz conjecture, based on combinatorial and modular criteria. In particular, the approach shows that the information contained in the numbers congruent to 1 and 3 mod 4 is sufficient to describe the entire dynamics of the function f(N).

6 Conclusions

The custom Collatz sequence is not just an operational simplification, but a potential theoretical key to address one of the most famous open problems in elementary mathematics.

This way of proceeding in the custom sequence leads to the same results as the canonical Collatz sequence and does not depend on the initial number as it applies in the same way and leads to the same results on integers with few digits as on integers with an unlimited number of digits.
The example of the number N=27, compared between canonical succession and custom sequence, shows that the reduction of iterations does not alter the final result, but optimizes its path.

Many tests have been carried out on different numbers containing even a number of numerous digits using a python program of my creation that faithfully follows my custom sequence with numbers having many digits that have confirmed the veracity of the conjecture.
Table 1 in column 10 shows that in each iteration **the difference between the total of even steps (which I indicate with *bi)* and the total of odd steps (which I indicate with *ai)* always increases by 1 unit or by 2 units in the next iteration** and this always produces a difference between *bi* and *ai* in subsequent iterations which always increases in each iteration and leads to the decrease of the values of N and the convergence of N to the number 1 when the following relation is respected. (see $\frac{3^{a_i} * N}{2^{b_i}} < 1$**paragraph 6 Proof of the Collatz conjecture of reference [1]**).

References

[1] Ardito prof. Nicola - Collatz's conjecture from an elementary point of view – https://independent.academia.edu/NICOLAARDITO1

[2] Eric Roosendaal - http://www.ericr.nl/wondrous/index.html 3x+1 problem

Contacts

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https://archive.org/search?query=collatz+ardito+nicola